On the Deformation of Thin Elastic Wires.

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1. An account of the various attempts which have been made to construct a theory of the deformation of a thin elastic wire, together with the solution of various problems of interest, will be found in the second volume of Mr. Love's recent Treatise on Elasticity. The above work also contains a variety of *geometrical* investigations connected with this subject, and the methods employed are of considerable novelty, power and elegance. But Mr. Love's treatment of the *physical* portion of the subject is not at all so satisfactory; and this is in great measure due to the fact, which I have commented upon in my recent paper on the Deformation of Thin Elastic Plates and Shells,* that he appears to entertain some objection against the method of expansion, and has also been unable to emancipate himself from the imperfect methods of the German and French schools.

* Amer. Journ. of Math., Vol. XVI, p. 255.

I am of opinion that the most satisfactory way of constructing a complete theory of the small deformations of thin wires is to employ the method explained in my paper on the Theory of Elastic Wires;* but unfortunately that investigation contains a slight slip in the work, which arose from my having copied an equation wrongly and used the wrong equation in a subsequent portion of the paper. In consequence of this, the values of the two flexural couples are not proportional to the changes of curvature, as ought to be the case. This circumstance may possibly have led Mr. Love to entertain doubts as to the soundness of the principles upon which the theory was based; whereas the real fact is that the theory is a perfectly sound and unimpeachable one, and when the error is corrected it leads to results which have been established by methods of a more or less imperfect character, which agree with those obtained by Mr. Love, and are now generally admitted to be correct.

Under these circumstances I think that a further exposition of the theory of wires is needed, and this is what I propose to give in the present paper. I shall commence with the theory of the small deformations of a naturally curved wire; I shall then discuss the theory of finite deformations, in which finite changes of curvature and twist occur; and I shall lastly work out the solutions of various problems of interest.

In most problems of practical interest, the wire is made of flexible and well-tempered metal such as steel; also its cross-section is uniform and circular, and the radius of the latter is small in comparison with the radius of principal curvature of the wire at any point of its length. Wires of this description are called *thin wires*, and to such wires the following investigation will be exclusively confined. It is also obvious that the central axis of a metal wire may usually be regarded as inextensible, since any extension of the axis which might possibly be produced by any given forces is extremely small in comparison with the flexion and torsion actually produced. We shall therefore suppose that the extension of the central axis may be neglected.

The General Equations of Equilibrium.

2. When a thin wire, whose natural form is any curve, is deformed, the lines which before deformation coincided with the principal normal, the binormal and the tangent to the central axis will not usually coincide after deformation with the principal normal, the binormal and the tangent to the deformed central axis;

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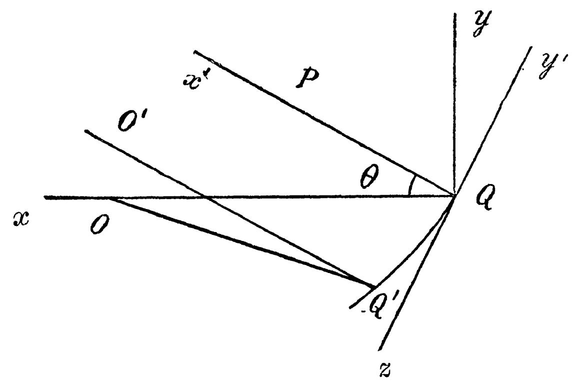
* Proc. Lond. Math. Soc., Vol. XXIII, p. 105.

and as our object is to obtain equations which are applicable to a wire finitely deformed, it is convenient to choose as our axis of reference the above-mentioned lines in the deformed wire. If, however, the deformation is small, it is immaterial whether the axes are supposed to refer to the deformed or the undeformed central axis, since any error which might be introduced would be of the second order of small quantities.

The resultant stresses which act across any transverse section of the deformed wire are six in number, and consist of

T = a tension along the tangent to the central axis,
N_1 = a shearing stress along the principal normal,
N_2 = a shearing stress along the binormal,
H = a torsional couple about the tangent,
G_1 = a flexural couple about the principal normal,
G_2 = a flexural couple about the binormal.

To obtain the equations of equilibrium, let Q be any point on the central axis; Qx, Qy, Qz the principal normal, binormal and tangent to the central axis at Q. Let Q' be any point on the central axis near Q; O, O' the centres of principal curvature at Q and Q'; let $\delta\phi$, $\delta\eta$ be the angles of contingence and torsion at Q, so that $QOQ' = \delta\phi$, $OQ'O' = \delta\eta$; let ρ, σ be the radii of principal curvature and torsion at Q. Also let $T + \delta T$, $N_1 + \delta N_1$, etc., be the values of the resultant stresses at Q', and $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$; $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ the components of the bodily forces and couples per unit of length of the wire.



The equations of equilibrium are obtained by resolving all the forces and couples which act upon the element δs, parallel to Qz, Qx, Qy. We thus obtain

$$\left.\begin{aligned}
\frac{dT}{ds} - \frac{N_1}{\rho} + \mathfrak{T} &= 0, \\
\frac{dN_1}{ds} - \frac{N_2}{\sigma} + \frac{T}{\rho} + \mathfrak{X} &= 0, \\
\frac{dN_2}{ds} + \frac{N_1}{\sigma} + \mathfrak{Y} &= 0, \\
\frac{dH}{ds} - \frac{G_1}{\rho} + \mathfrak{N} &= 0, \\
\frac{dG_1}{ds} - \frac{G_2}{\sigma} + \frac{H}{\rho} - N_2 + \mathfrak{L} &= 0, \\
\frac{dG_2}{ds} + \frac{G_1}{\sigma} + N_1 + \mathfrak{M} &= 0.
\end{aligned}\right\} \quad (1)$$

In statical problems the three couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are usually zero; but in dynamical problems they must be replaced by the time variations (taken with the negative sign) of the components of the angular momentum of the element.

Equations (1) in their present form do not enable us to solve any statical or dynamical problems; in order to do this we require the values of the three couples. We shall hereafter show that the flexural couples are proportional to the changes of curvature, whilst the torsional couple is proportional to the change of twist; and these theorems combined with (1) are sufficient to enable us to solve a variety of problems relating to finite deformations. When the deformation is small, the values of the three couples (and consequently the changes of curvature and twist) can be expressed in terms of the displacements of the point Q, together with a certain angle β which is connected with the twist; and these results combined with the condition of inextensibility will furnish a sufficient number of equations for the solution of every problem.

Theory of Small Deformations.

3. In the theory of thin plates and shells, the three stresses R, S, T vanish at the free surfaces of the shell, *provided the latter are not subjected to any surface pressures or tangential stresses;* and I have shown in my previous papers that, subject to this limitation, the terms of lowest order which these stresses contain are quadratic functions of h and h', where $2h$ is the thickness of the shell and h' is the distance of a point in its substance from the middle surface. The coefficients of h and h' in these quadratic functions are unknown quantities which

cannot be determined by any direct method; but I have shown that if the investigation is confined to an approximate solution which does not involve higher powers of the thickness than the cube, it is not necessary to ascertain the values of these unknown quantities; in other words, *the three stresses which vanish at the surface may be treated as zero.* Under these circumstances it appeared to me that the most natural and appropriate course was to employ a similar hypothesis as the basis of the theory of wires.

In the figure let P be *any* point in the cross-section through Q, and let Qx', Qy', Qz be a subsidiary set of rectangular axes of which the axis of x' passes through P; also let P, Q, R, S, T, U be the six components of stress at P referred to this subsidiary set of axes. Since the cross-section of the wire is circular, the three stresses P, T, U must vanish at the surface provided the latter is free from stress; and from analogy to the theory of thin plates it is natural to suppose that within the substance of the wire these stresses are *small quantities which may be treated as zero* provided the solution is confined to a certain degree of approximation. And I found that if the terms of lowest order in P, T, U were quadratic functions of c and r, where c is the radius of the cross-section and $QP = r$, the values of the three couples could be calculated as far as the *fourth* power of the radius by treating P, T, U as zero, since the retention of these quantities would lead to terms involving higher powers of c than the fourth. I accordingly based the theory on the following fundamental hypothesis:

The three stresses P, T, U are small quantities which may be treated as zero, provided the surface of the wire is not subjected to any surface forces such as pressures or tangential stresses; and provided also that the approximate expressions for the energy and couples do not include any higher powers of the radius of the cross-section than the fourth.

This hypothesis may possibly appear a bold one, especially as I was unable to bring forward in support of it evidence furnished by the general equations of elasticity of the same character as can be produced in the case of the corresponding hypothesis which forms the foundation of the theory of thin plates and shells; but the results to which this hypothesis lead conclusively establish its correctness.

4. The development of the theory of wires has been retarded by an erroneous assumption of Saint-Venant, that the three stresses P, Q, U are *accurately* zero.

Saint-Venant made this hypothesis the basis of his theory of the torsion of prisms, and it is remarkable that he was thereby led to results which are undoubtedly correct when the prism is *infinitely long.* I have considered this theory in my paper on wires,* and have shown on page 125 that all the results can be obtained without the aid of this highly objectionable hypothesis. The fallacy of writers who have followed Saint-Venant lies in the fact that they have imagined that a hypothesis which happens to be true in a class of problems of a very special character, can be made the basis of a general theory of wires. It can be shown that when the cross-section is circular, Q will vanish to a certain order of approximation provided P does; consequently if P may be treated as zero, Q may also be so treated. But a result which drops out incidentally in the course of the work is a totally different thing from an assumption which dogmatically asserts that the result is true; and the objection to assuming that Q may be treated as zero lies in the fact that, since it does not vanish at the boundary, no valid reason can be assigned for supposing that in the interior of the wire it is a small quantity which may be neglected.

The fact that the stress Q may not in general be treated as zero, unless the cross-section is circular, may be seen by considering the case of a wire of elliptic cross-section. If the ellipse be supposed to degenerate into two infinite parallel straight lines, the wire will become a thin plate, and the stress P in the theory of wires becomes the stress R in the theory of plates; whilst the stress Q in the theory of wires becomes the stress P (or Q) in the theory of plates. Since P (or Q) in the theory of plates may not be treated as zero, it follows that the stress Q in the theory of wires may not in general be so treated.

5. The mathematical development of the fundamental hypothesis and the procedure employed for calculating the values of the three couples are so fully explained on pages 108 to 116 of my paper on the Theory of Wires previously referred to, that it will be unnecessary to reproduce the investigation. I shall therefore proceed to show how the error I have alluded to arose, and how it is to be corrected.

The strain g is correctly given by equation (11) of that paper, but in copying out equations (13) to (18) on p. 111 a term has been omitted in (15). The correct equation is

$$g = \frac{1}{\rho - r\cos\theta}\left(\frac{dw'}{d\phi} - \frac{\rho}{\sigma}\,\frac{dw'}{d\theta} - u'\cos\theta + v'\sin\theta\right). \tag{2}$$

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* Proc. Lond. Math. Soc., Vol. XXIII.

The value of w' is correctly given by equation (31), and when the omitted term is supplied in equation (32) it will be found to lead to exactly the same values of the two flexural couples as those given by Mr. Love,* which are, as he has shown, proportional to the changes of curvature. The value of the torsional couple given by myself is quite correct, and it is proportional to the change of twist.

For brevity write

$$\lambda = \frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma} \quad , \quad \mu = \frac{dv}{ds} + \frac{u}{\sigma}, \tag{3}$$

$$\mathfrak{P} = \frac{d\lambda}{ds} - \frac{\mu}{\sigma} + \frac{\sigma_1 - \sigma_3}{\rho}, \quad \mathfrak{Q} = \frac{d\mu}{ds} + \frac{\lambda}{\sigma} - \frac{\beta}{\rho}, \tag{4}$$

where u, v, w are the displacements of a point on the central axis along the principal normal, binormal and tangent, and we shall obtain the following equations:

$$\left.\begin{aligned} u' &= r\sigma_1 + u\cos\theta + v\sin\theta + \frac{(m-n)\,r^2}{4m}(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \\ v' &= r\beta + v\cos\theta - u\sin\theta + \frac{(m-n)\,r^2}{4m}(\mathfrak{P}\sin\theta - \mathfrak{Q}\cos\theta), \end{aligned}\right\} \tag{5}$$

$$e = f = \sigma_1 + \frac{(m-n)\,r}{2m}(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \tag{6}$$

$$g = \sigma_3 - r(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \tag{7}$$

$$\sigma_1 = \sigma_2 = -\frac{m-n}{2m}\sigma_3. \tag{8}$$

From equations (6) and (7) it follows that $P = Q$, and since P has been assumed to be a small quantity which may be treated as zero to the order of approximation adopted, it follows that Q may also be treated as zero; accordingly Saint-Venant's assumption with regard to the latter quantity *drops out as an incidental result in the case of a wire of circular cross-section.*

From equations (6), (7) and (8) we obtain

$$e = f = -\frac{m-n}{2m}g. \tag{9}$$

This result shows that for *any fibre* which is parallel to the axis of the wire, *the ratio of lateral contraction to longitudinal elongation is equal to Poisson's ratio.* Equation (8) only establishes this proposition for the central fibre.

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* Theory of Elasticity, Vol. II, pp. 168–169.

Again, if R be the normal traction at *any point* of the cross-section,

$$R = (m+n)\,g + (m-n)(e+f),$$
$$= \frac{(3m-n)\,n}{m}\,g = qg, \tag{10}$$

by (9), where q is Young's modulus. We have, therefore, proved the following theorem:

When a wire of circular cross-section is twisted as well as bent, the normal traction at any point of a cross-section is equal to the product of Young's modulus and the extension at that point.

6. The values of the two flexural couples are

$$\left.\begin{aligned} G_1 &= \int_0^c\int_0^{2\pi} Rr^2 \sin\theta\, dr\, d\theta, \\ G_2 &= -\int_0^c\int_0^{2\pi} Rr^2 \cos\theta\, dr\, d\theta. \end{aligned}\right\} \tag{11}$$

Now we have stated that in most practical applications the extension of the central axis may safely be neglected; under these circumstances $\sigma_3 = 0$, whence if we substitute the value of R in (11) from (10) and (7), and the values of $\mathfrak{P}$ and $\mathfrak{Q}$ from (4), we shall obtain

$$\left.\begin{aligned} G_1 &= -\tfrac{1}{4}\pi c^4 q\left(\frac{d\mu}{ds} + \frac{\lambda}{\sigma} - \frac{\beta}{\rho}\right), \\ G_2 &= \quad \tfrac{1}{4}\pi c^4 q\left(\frac{d\lambda}{ds} - \frac{\mu}{\sigma}\right), \end{aligned}\right\} \tag{12}$$

which may be written by virtue of (3),

$$\left.\begin{aligned} G_1 &= \tfrac{1}{4}\pi c^4 q\left\{\frac{\beta}{\rho} - \frac{d}{ds}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma}\right)\right\}, \\ G_2 &= \tfrac{1}{4}\pi c^4 q\left\{\frac{d}{ds}\left(\frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right)\right\}, \end{aligned}\right\} \tag{13}$$

which agree with the expressions for the flexural couples obtained by Mr. Love.

The value of the torsional couple H as shown in my paper is

$$H = \tfrac{1}{2}\pi c^4 n\left(\frac{d\beta}{ds} + \frac{1}{\rho}\,\frac{dv}{ds} + \frac{u}{\rho\sigma}\right). \tag{14}$$

Energy of the Wire.

7. The potential energy of the deformed wire per unit of length is

$$\tfrac{1}{2}\int\!\!\int [(m+n)\Delta^2 + n\{a^2+b^2+c^2-4(ef+fg+ge)\}]\,\rho^{-1}(\rho - r\cos\theta)\,r\,dr\,d\theta. \quad (1)$$

By the fundamental hypothesis the strains b and c are to be neglected, since on integration they would lead to terms involving higher powers of the radius of the cross-section than the fourth. The value of the strain a is shown in my paper to be

$$a = \frac{r}{\rho}\left(\frac{d\beta}{d\phi} + \frac{1}{\rho}\frac{dv}{d\phi} + \frac{u}{\sigma}\right) = r\left(\frac{d\beta}{ds} + \frac{\mu}{\rho}\right).$$

The values of e, f and g given by (6) and (7) of §5 only include the first power of r, and in order to calculate the potential energy when the central axis is supposed to undergo extension, it would be necessary to proceed to a higher degree of approximation so as to obtain the terms in r^2; for if e contained the term $\mathfrak{P}r^2$, the expression (1) would contain a term $\int\!\!\int \mathfrak{P}\sigma_1 r^3\,dr\,d\theta$ which is proportional to the fourth power of the radius. If, however, the central line is supposed to be inextensible, the terms σ_1, σ_3 are zero, and the expression for the potential energy becomes

$$W = \tfrac{1}{8}\pi c^4\left\{2n\left(\frac{d\beta}{ds} + \frac{\mu}{\rho}\right)^2 + q\left(\frac{d\lambda}{ds} - \frac{\mu}{\sigma}\right)^2 + q\left(\frac{d\mu}{ds} + \frac{\lambda}{\sigma} - \frac{\beta}{\rho}\right)^2\right\}. \quad (2)$$

Recollecting the values of the couples given by (13) and (14) of §5, and putting A and C for the flexural and torsional rigidities, this may be written

$$W = \tfrac{1}{2}(G_1^2/A + G_2^2/A + H^2/C), \quad (3)$$

a form which is often useful.

The kinetic energy $\mathfrak{T}$ per unit of length is given by the equation

$$\mathfrak{T} = \tfrac{1}{2}\pi hc^2(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + \tfrac{1}{4}\pi hc^4\dot{\beta}^2 + \tfrac{1}{8}\pi hc^4(\dot{\lambda}^2 + \dot{\mu}^2), \quad (4)$$

where h denotes the mass of a unit of length.

8. These formulæ may be verified by means of the variational equation of motion, which thus forms a test of the correctness of the work and of the fundamental hypothesis on which the theory is based.

The equation in question is

$$\delta W + \delta\mathfrak{T} = \delta U + \delta\mathfrak{L},$$

where

$$\delta\mathfrak{T} = \pi hc^2 \int \{\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w + \tfrac{1}{2}c^2\ddot{\beta}\delta\beta + \tfrac{1}{4}c^2(\ddot{\lambda}\delta\lambda + \ddot{\mu}\delta\mu)\}\, ds,$$

$$\delta U = \int (X\delta u + Y\delta v + Z\delta w)\, ds,$$

$$\delta\mathfrak{L} = G_2\delta\lambda - G_1\delta\mu + H\delta\beta + N_1\rho\frac{d\delta w}{ds} + N_2\delta v + T\delta w,$$

and if we work out the variation by the ordinary methods of the Calculus of Variations, and take account of the condition of inextensibility, we shall find that (i) we shall reproduce the values of the three couples which we have already obtained; (ii) we shall reproduce the third of equations (1) of §2; (iii) we shall reproduce an equation which is the result of eliminating T between the first and second of (1).

Theory of Finite Deformations.

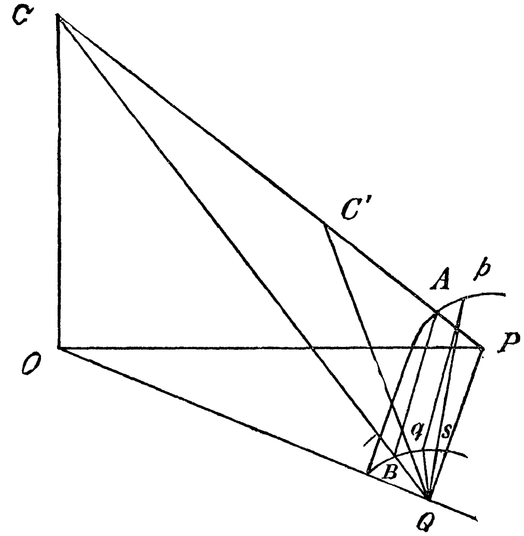
9. When a thin wire is slightly deformed and the central axis undergoes no extension, the expressions for the flexural and torsional couples are given by (13) and (14) of §5; and it is shown in Mr. Love's treatise that the expressions in brackets in these equations are respectively equal to the changes of curvature and twist. It may therefore be anticipated that these couples are proportional to the same geometrical quantities when the deformation is not small; and this we shall now show to be the case.

The method of constructing a theory of finite deformations, which has been adopted by Mr. Love in the second volume of his book, appears to me to be very unsatisfactory and difficult to follow. I do not find the argument on p. 93 at all convincing, and on p. 157 he does not attempt to give any formal proof that these three couples are proportional to the changes of curvature and twist, but dismisses the subject with the perfunctory remark that "as there is some controversy about this result, it may be as well to indicate another method of proof," which occupies a paragraph of about a dozen lines. It was no doubt unfortunate that, owing to the slip which I have corrected in the previous part of this paper, I failed to obtain the correct values of the two flexural couples; but surely the author of what purports to be a classical treatise on Elasticity ought to have cleared up this point and not to have left it in doubt.

10. If we fix our attention on a small element of a finitely deformed wire, whose centre of inertia is P, the displacement of any point Q of the element

will be made up of three distinct parts, viz. (i) a translation of the whole element from one position in space to another, (ii) a rotation of the whole element about some axis through P, (iii) a distortion of the element which produces an alteration in the position of different points of its substance relatively to one another. The first two parts are displacements such as a rigid body might undergo, and cannot therefore produce any strain in the element. The internal strains which the element experiences accordingly depend solely on the third kind of displacement, that is, on the displacement of Q relatively to P.

Let PQ be the central axis of the undeformed element; draw any plane through the tangent at P, let C be the centre of curvature of the element in this plane, and O the centre of principal curvature.



The bending can obviously be effected by means of two opposite couples applied to the ends of the element, whose axes are perpendicular to the plane PCQ, which will be called the *plane of bending*. The effect of the bending will be to change the curvature at P in the plane PCQ, and also to extend or contract all filaments, such as pq, which are parallel to PQ, but the lines pq and PQ will remain sensibly parallel after bending.

The torsion can be effected by applying equal and opposite couples to the ends of the element, whose axes coincide with the tangents at P and Q. The effect of the torsion will be to twist all lines such as pq which lie on the surface of the cylinder $ApqB$ through small angles, so that after torsion they will assume positions slightly inclined to their former ones.

The flexion and torsion will also produce various deformations of a secondary character, but these may be neglected in working out the approximate solution which we shall obtain.

Let ρ, ρ' be the radii of curvature in the plane of bending before and after deformation, ρ_1 the radius of principal curvature; also let $qQO = \theta$, $qQC = \theta'$, $Qq = r$.

Before bending,

$$\frac{pq}{PQ} = \frac{\rho_1 - r\cos\theta}{\rho_1}.$$

The effect of the bending will be to displace the point q through a small space $r\delta\omega\cos\theta'$, where $\delta\omega$ is the rotation due to bending about a line through Q perpendicular to the plane of bending. Now if C' be the centre of curvature in the plane PCQ after bending,

$$\delta\omega = PC'Q - PCQ = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right),$$

whence the displacement of q along pq is

$$r\delta\omega\cos\theta' = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta',$$

consequently if σ_3' be the extension,

$$\sigma_3' = -\frac{r\delta\omega\cos\theta'}{pq} = -\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta', \tag{1}$$

if higher powers of r than the first be neglected.

The effect of the torsion will be to displace the line pq to the position ps, and therefore the above expression for the extension is not rigorously accurate when there is torsion as well as flexion, but the error depends upon the square of the small angle qQs and may be neglected.

If R' be the normal traction perpendicular to the cross-section, we have already proved that *when the deformation is small,*

$$R' = q\sigma_3', \tag{2}$$

where q is Young's modulus; and since our results in the present case must be consistent with those which we have already obtained when the deformation is small, *we shall assume that* (2) *is true when the deformation is finite.* This is the only assumption which it will be necessary to make. Under these circumstances we obtain from (1) and (2),

$$R' = -q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta'.$$

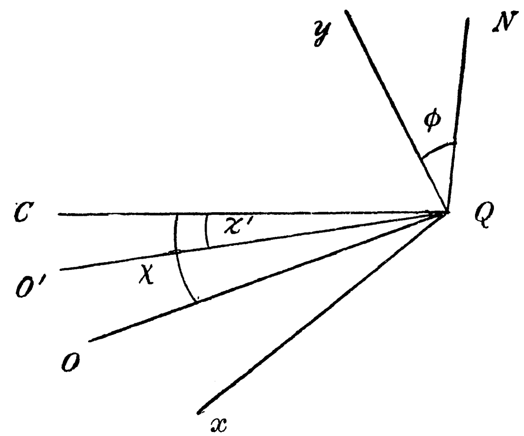
The flexural couple G about the normal through Q to the plane of bending is

$$G = -\int_0^c \int_0^{2\pi} R' r^2 \cos^2\theta' dr\, d\theta',$$

$$= \tfrac{1}{4}\pi c^4 q \left(\frac{1}{\rho'} - \frac{1}{\rho}\right), \tag{3}$$

and is therefore proportional to the change of curvature in the plane of bending. The flexural couple about QC is obviously zero.

11. We shall now resolve this couple about two arbitrary axes Qx, Qy at right angles to one another in a plane perpendicular to the tangent at Q.



In the figure, QC is the normal to the wire in the plane of bending, QN is the normal to this plane at Q, and C is the centre of curvature in this plane. Let $CQx = NQy = \phi$; then if G_x, G_y be the flexural couples about Qx, Qy, and A the flexural rigidity,

$$G_x = -G\sin\phi = -A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\sin\phi, \tag{4}$$

$$G_y = \quad G\cos\phi = \quad A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\cos\phi. \tag{5}$$

Let QO, QO' be the principal normals at Q before and after bending; O, O' the centres of principal curvature; also let $CQO = \chi$, $CQO' = \chi'$. Let R_x, R'_x, R_y, R'_y be the radii of curvature before and after bending in planes perpendicular to Qx, Qy, and let ρ_1, ρ'_1 be the radii of principal curvature

before and after bending. Then

$$\left.\begin{aligned}
\frac{1}{\rho'} &= \frac{1}{\rho_1'}\cos\chi' , & \frac{1}{\rho} &= \frac{1}{\rho_1}\cos\chi , \\
\frac{1}{R_x'} &= \frac{1}{\rho_1'}\sin(\phi-\chi'), & \frac{1}{R_x} &= \frac{1}{\rho_1}\sin(\phi-\chi), \\
\frac{1}{R_y'} &= \frac{1}{\rho_1'}\cos(\phi-\chi'), & \frac{1}{R_y} &= \frac{1}{\rho_1}\cos(\phi-\chi).
\end{aligned}\right\} \quad (6)$$

Since the curvature in the plane through the tangent which is perpendicular to the plane of bending is unchanged,

$$\frac{1}{\rho_1'}\sin\chi' = \frac{1}{\rho_1}\sin\chi. \quad (7)$$

From the first and second of (6) combined with (7) we get

$$\frac{1}{R_x'} = \frac{1}{\rho_1}\sin\phi - \frac{1}{\rho_1}\cos\phi\sin\chi,$$

$$\frac{1}{R_x} = \frac{1}{\rho}\sin\phi - \frac{1}{\rho_1}\cos\phi\sin\chi,$$

whence

$$\frac{1}{R_x'} - \frac{1}{R_x} = \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\sin\phi,$$

accordingly

$$G_x = -A\left(\frac{1}{R_x'} - \frac{1}{R_x}\right), \quad (8)$$

and in the same way

$$G_y = \quad A\left(\frac{1}{R_y'} - \frac{1}{R_y}\right), \quad (9)$$

where $A = \frac{1}{4}\pi q c^4$ is the flexural rigidity. This shows that the flexural couples about the normals to any two planes at right angles to one another are proportional to the changes of curvature in those planes. The negative sign in (8) is accounted for by the fact that owing to the way in which the quantities are measured G_x is positive when the curvature is diminished.

12. We must now find the torsional couple.

The flexion simply displaces the point q along pq; the torsion produces a displacement along the circular arc to s, so that the line pq assumes the position ps.

Let the angles

$$qps = \psi, \quad qQs = \tau . PQ,$$

then
$$r\tau . PQ = ps . \psi,$$
whence
$$\psi = r\tau . \frac{PQ}{pq} = \frac{\rho r\tau}{\rho - r\cos\theta}.$$

Now ψ is the shearing strain perpendicular to Qq in the plane QqB, whence if H be the torsional couple,

$$\begin{aligned} H &= n\int_0^c \int_0^{2\pi} \psi r^2 dr\, d\theta \\ &= \tfrac{1}{2}\pi c^4 n\tau. \end{aligned} \tag{10}$$

The quantity τ is the change of twist, and is the same quantity which Mr. Love denotes by $\tau' - \tau$.

Potential Energy.

13. Since the work done by a stress is equal to half the product of the stress into the strain produced, it follows that the work done by flexion is

$$\tfrac{1}{2}\int_0^c \int_0^{2\pi} R'\sigma_3' r\, dr\, d\theta = \tfrac{1}{2} q \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2 \int_0^c \int_0^{2\pi} r^3 \cos^2\theta\, dr\, d\theta = \tfrac{1}{8}\pi c^4 q \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2$$

by (1) and (2).

The work done by torsion is

$$\tfrac{1}{2} n \int_0 \int_0^{2\pi} \psi^2 r\, dr\, d\theta = \tfrac{1}{4}\pi c^4 n\tau^2.$$

It therefore follows that if W be the potential energy per unit of length,

$$W = \tfrac{1}{4}\pi c^4 \left\{ \tfrac{1}{2} q \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2 + n\tau^2 \right\}, \tag{11}$$

which agrees with the results we have already obtained when the deformation is small.

Equilibrium of Naturally Straight Wires.

14. The preceding formulæ can be simplified when the wire is naturally straight. In this case the curvature in every plane through the axis of the wire is zero before deformation; and since the change of curvature in that plane through the tangent to the deformed wire which is perpendicular to the plane of bending is zero after deformation, it follows that the curvature in the above-mentioned plane is also zero after deformation. Hence the plane of bending is the osculating plane of the deformed wire.

From this it follows that

$$G_1 = 0, \quad G_2 = A/\rho,$$

whence, by the fourth of equations (1) of §2,

$$\frac{dH}{ds} = 0,$$

or

$$H = \text{const.},$$

which shows that the torsional couple is constant throughout the length of the wire.

This is a very important proposition.

15. We shall now proceed to integrate the equations of equilibrium of a naturally straight wire.

Since H is constant and $G_1 = 0$, it follows that if ϖ denote the curvature so that $\varpi = 1/\rho$, equations (1) of §2 become

$$\frac{dT}{ds} - N_1\varpi = 0, \tag{1}$$

$$\frac{dN_1}{ds} - \frac{N_2}{\sigma} + T\varpi = 0, \tag{2}$$

$$\frac{dN_2}{ds} + \frac{N_2}{\sigma} = 0, \tag{3}$$

$$\frac{G_2}{\sigma} - H\varpi + N_2 = 0, \tag{4}$$

$$\frac{dG_2}{ds} + N_1 = 0. \tag{5}$$

Since $G_2 = A\varpi$, we obtain from (1) and (5)

$$\frac{dT}{ds} + A\varpi\frac{d\varpi}{ds} = 0,$$

whence

$$T = P - \tfrac{1}{2}A\varpi^2, \tag{6}$$

where P is a constant.

From (4) we get

$$N_2 = \left(H - \frac{A}{\sigma}\right)\varpi, \tag{7}$$

and from (1) and (6)

$$N_1 = -A\frac{d\varpi}{ds}. \tag{8}$$

From (7) and (8) combined with (3) we get

$$H\varpi\frac{d\varpi}{ds} - A\frac{d}{ds}\left(\frac{\varpi^2}{\sigma}\right) = 0,$$

whence
$$\frac{A}{\sigma} = \tfrac{1}{2}H + \frac{Q}{\varpi^2}, \tag{9}$$

where Q is a constant; accordingly by (7)
$$N_2 = \left(\tfrac{1}{2}H - \frac{Q}{\varpi^2}\right)\varpi. \tag{10}$$

To obtain a third integral, substitute the values of T, N_1, N_2 from (6), (8) and (10) in (2) and we get
$$A^2\frac{d^2\varpi}{ds^2} + (\tfrac{1}{4}H^2 - PA)\varpi - \frac{Q^2}{\varpi^3} + \tfrac{1}{2}A^2\varpi^3 = 0. \tag{10.A}$$

Integrating we obtain
$$\left(A\varpi\frac{d\varpi}{ds}\right)^2 = -\tfrac{1}{4}A^2\varpi^6 + (AP - \tfrac{1}{4}H^2)\varpi^4 + R\varpi - Q^2, \tag{11}$$

where R is another constant.

16. From (11) we see that $(d\varpi^2/ds)^2$ is a cubic function of ϖ^2, and therefore ϖ^2 can be expressed in terms of s by means of elliptic functions of the first kind. Let $\frac{1}{4}A^2Z$ denote this cubic function; then collecting our results from (6), (9) and (11), we have the following three first integrals of the equations of equilibrium, viz.
$$\left.\begin{aligned} T &= P - \tfrac{1}{2}A\varpi^2, \\ \frac{A}{\sigma} - \tfrac{1}{2}H &= \frac{Q}{\varpi^2}, \\ \frac{d\varpi^2}{ds} &= Z^{\frac{1}{2}}, \end{aligned}\right\} \tag{12}$$

The first of (12) merely determines the tension, but the second leads to important results. If the curve assumed by the wire is a plane curve, $\sigma = \infty$; whence if Q is not zero, ϖ must be constant, and therefore the curve is a circle. If, however, Q is zero, σ is constant, and therefore the curve assumed by the wire is one of constant tortuosity; and if we suppose the curve to be plane, so that $\sigma = \infty$, it follows that H must be zero and the wire devoid of twist. From these results it follows that if a naturally straight wire is twisted as well as bent, the circle is the only plane curve which is a possible figure of equilibrium; but if the wire is bent without being twisted, a family of plane curves exist whose curvature is expressed in terms of the arc by means of the last of (12). The

curves are of course the elastica family, whose properties have been discussed by various writers.*

17. We shall now proceed to integrate (11). Writing $\varpi^2 = z$, the equation becomes

$$\left(\frac{dz}{ds}\right)^2 = -z^3 + \left(\frac{4P}{A} - \frac{H^2}{A^2}\right)z^2 + \frac{4R}{A^2}z - \frac{4Q^2}{A^2}. \tag{13}$$

The form of this equation shows that the right-hand side is equivalent to

$$(z+\alpha)(z-\beta)(\gamma-z) = -z^3 + z^2(\beta+\gamma-\alpha) + z(\alpha\beta+\alpha\gamma-\beta\gamma) - \alpha\beta\gamma, \tag{14}$$

and we must now discuss the possible values of α, β, γ.

(i) Let α be real and positive; then if β and γ are real they must both be of the same sign, and this sign must be positive, otherwise $(dz/ds)^2$ would be negative, which is impossible. It also follows that $\gamma > z > \beta$.

If β and γ were complex, we should have

$$(z-\beta)(\gamma-z) = -(z-p-\iota q)(z-p+\iota q),$$

which would make $(dz/ds)^2$ negative.

(ii) Let α be real and negative; then if we suppose β is real and negative we fall back on the previous case with α and β interchanged. But if α is real and negative and β is real and positive, we must have γ real and negative, so that, writing $-\alpha$, $-\gamma$ for α and γ, the left-hand side of (12) becomes $(\alpha - z)(z-\beta)(z+\gamma)$, which is the first case with α and γ interchanged.

It is also impossible for α to be real and negative and β and γ complex, for this would make the cubic expression complex, since we should have to write $\beta = p + \iota q$, $\gamma = -(p - \iota q)$.

(iii) Let α be a complex of the form $p + \iota q$, then if γ is real and positive, β must be a complex of the form $p - \iota q$, since the product $\alpha\beta$ must be real and positive; but in this case

$$(z+\alpha)(z-\beta) = (z+p+\iota q)(z-p+\iota q),$$

which is complex.

If γ is real and negative, β must be of the form $-(p-\iota q)$, in which case

$$(z+\alpha)(z-\beta) = (z+p+\iota q)(z+p-\iota q) = (z+p)^2 + q^2,$$

which makes $(dz/ds)^2$ negative.

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* A very complete account of the elastica will be found in Halphen's *Traité des Fonctions Elliptiques*, Vol. II, Ch. V.

(iv) Let α be complex, and β positive; then γ must be of the form $p - \iota q$, so that

$$(z+\alpha)(\gamma - z) = -(z+p+\iota q)(z-p+\iota q),$$

which is complex. If β is negative, we must have $\gamma = -(p - \iota q)$, so that

$$(z+\alpha)(\gamma - z) = -(z+p+\iota q)(z+p-\iota q),$$

which makes $(dz/ds)^2$ negative.

We therefore conclude that the only possible case to consider arises when α, β, γ are all real and positive, and $\gamma > \beta$.

The expression to be integrated now becomes

$$ds = \frac{dz}{\{(z+\alpha)(z-\beta)(\gamma - z)\}^{\frac{1}{2}}}.$$

Let

$$u = (z+\alpha)^{\frac{1}{2}},$$

then

$$ds = \frac{2du}{(u^2 - \alpha - \beta)^{\frac{1}{2}}(\gamma + \alpha - u^2)^{\frac{1}{2}}}.$$

Writing

$$a^2 = \alpha + \gamma, \quad b^2 = \alpha + \beta,$$

we obtain

$$ds = \frac{2du}{(u^2 - b^2)^{\frac{1}{2}}(a^2 - u^2)^{\frac{1}{2}}}.$$

In this write

$$u^2 = a^2 \cos^2\phi + b^2 \sin^2\phi,$$

and we get

$$\tfrac{1}{2}\, ds = -\frac{d\phi}{\{a^2 - (a^2 - b^2)\sin^2\phi\}^{\frac{1}{2}}},$$

giving

$$u = a\,\mathrm{dn}\,(\tfrac{1}{2}as + \mathfrak{A}), \quad k^2 = (a^2 - b^2)/a^2, \tag{15}$$

where $\mathfrak{A}$ is the constant of integration. We therefore finally obtain

$$\varpi^2 = (\alpha+\gamma)\,\mathrm{dn}^2\{\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s + \mathfrak{A}\} - \alpha, \tag{16}$$

$$k^2 = \frac{\gamma - \beta}{\gamma + \alpha}. \tag{17}$$

The constant $\mathfrak{A}$ may be put equal to zero if s be measured from the point where $\varpi = \sqrt{\gamma}$.

18. The greatest value of $\mathrm{dn}x$ occurs when $x = 0$ and the least when $x = K$, and since $\gamma > \beta$, it follows that the maxima values of ϖ^2 occur when $\frac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s = 2nK$, in which case ϖ is equal to $\sqrt{\gamma}$, and the minima when $\frac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s = (2n+1)K$ when it is equal to $\sqrt{\beta}$. Hence the curve cannot have any points of inflexion unless $\beta = 0$.

We shall now suppose the wire to form a closed curve, and that there are n maxima and minima values of ϖ; if l be its length, we must have

$$K = \tfrac{1}{4}(\alpha + \gamma)^{\frac{1}{2}} l/n.$$

Now the least value of K is $\frac{1}{2}\pi$, whence the above equation requires that

$$n < l(\alpha + \gamma)^{\frac{1}{2}}/2\pi,$$

and consequently the number of maxima and minima cannot be greater than the integer which is nearest to $l(\alpha + \gamma)^{\frac{1}{2}}/2\pi$. It also follows that there are no points of inflexion since ϖ^2 can never vanish.

19. The conditions of the problem, as we have already shown, require that α, β, γ should all be positive and that $\gamma > \beta$. It is however possible for α or β to be zero, and we have accordingly two special cases to consider. These particular results may of course be deduced from the general one, but it will be simpler to start from (11).

If α or β is zero, it follows that $Q = 0$, in which case the curve assumed by the wire is one of constant tortuosity; under these circumstances ϖ^2 cuts out from both sides of (11) and the equation becomes

$$4\left(\frac{d\varpi}{ds}\right)^2 = \frac{4R}{A^2} - \left(\frac{H^2}{A^2} - \frac{4P}{A}\right)\varpi^2 - \varpi^4. \tag{18}$$

Since $d\varpi/ds$ is essentially a real quantity, it follows that if $H^2/A > 4P$, R cannot be negative, and consequently if R is negative H^2/A must be less than $4P$, and we have therefore two cases to consider.

20. Case I. Let R be positive, and write

$$\left.\begin{aligned} a^2b^2 &= \frac{4R}{A^2}, \\ a^2 - b^2 &= \frac{H^2}{A^2} - \frac{4P}{A}, \end{aligned}\right\} \tag{19}$$

then (18) becomes

$$2\frac{d\varpi}{ds} = (a^2 + \varpi^2)^{\frac{1}{2}}(b^2 - \varpi^2)^{\frac{1}{2}},$$

which shows that this corresponds to the case of $\beta = 0$. Putting $\varpi = b\cos\phi$, we get

$$-2\frac{d\phi}{ds} = (a^2 + b^2 - b^2\sin^2\phi)^{\frac{1}{2}},$$

whence
$$\varpi = b \operatorname{cn} \tfrac{1}{2}(a^2 + b^2)^{\frac{1}{2}} s, \tag{20}$$
$$k = \frac{b}{(a^2 + b^2)^{\frac{1}{2}}}, \tag{21}$$

the constant being chosen so that $\varpi = b$ when $s = 0$.

Since ϖ vanishes and changes sign when

$$\tfrac{1}{2}(a^2 + b^2)^{\frac{1}{2}} s = (2n + 1) K,$$

it follows that the curve has points of inflexion.

21. Case II. Let R be negative. We must now write

$$\left.\begin{aligned} a^2 b^2 &= -\frac{4R}{A^2}, \\ a^2 + b^2 &= \frac{4P}{A} - \frac{H^2}{A^2}, \end{aligned}\right\} \tag{22}$$

and (18) becomes

$$2\frac{d\varpi}{ds} = (a^2 - \varpi^2)^{\frac{1}{2}}(\varpi^2 - b^2)^{\frac{1}{2}},$$

which corresponds to $\alpha = 0$ in the general case. The integral of this is

$$\left.\begin{aligned} \varpi &= a \operatorname{dn} \tfrac{1}{2} as, \\ k &= (a^2 - b^2)^{\frac{1}{2}}/a, \end{aligned}\right\} \tag{23}$$

and since ϖ can never vanish and change sign, there are no points of inflexion.

Stability of a Deformed Elastic Wire.

22. The stability of a deformed elastic wire may be investigated by three methods, which we shall proceed to explain.

23. The first method consists in supposing the wire to perform small oscillations about its configuration of equilibrium and finding their periods; the condition of stability is that the roots of the period equation should be real. This method possesses advantages when the periods are of acoustical interest; its chief defect is that it is somewhat indirect, and often leads to rather long and complicated expressions.

24. The second method, which has been employed by Prof. Greenhill* in considering the stability of a column under thrust and twist, depends upon some-

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* Proc. Inst. Mech. Engineers, 1883.

what refined considerations, and will be best illustrated by considering a special problem.

Let a naturally straight wire be bent and twisted and the ends joined together. It is easy to show, and will afterwards be proved, that a circle is a possible figure of equilibrium. Let us now assume that the circular form is stable when the torsional couple H is less than H_0, where H_0 is the quantity whose value we wish to determine. If H is very slightly greater than H_0, a figure of equilibrium will exist in which the wire assumes the form of a sinuous curve which differs very slightly from a circle, and will be derived therefrom by supposing the wire to undergo small displacements u, v, w and β. We must therefore solve the equations of equilibrium on the supposition that the sinuous form is a possible one when the torsional couple has an arbitrary value H. Since u, v, w and β must be periodic with respect to the vectorial angle ϕ, each of these quantities must be proportional to $\varepsilon^{\iota s \phi}$, where s is any integer greater than unity, since $s = 0$, $s = 1$ correspond to rigid body displacements which can produce no alteration in the state of strain; and we shall thus be led to an equation of the form

$$H = F(s).$$

When H is less than the minimum value of $F(s)$, if such exist, it will be impossible to satisfy this equation, and consequently equilibrium in the sinuous form cannot exist, from which it follows that equilibrium in the circular form is stable. But when H is slightly greater than the minimum value of $F(s)$, equilibrium in the sinuous form is possible, and the precise form of the curve can be determined by means of a Fourier's series. If H_0 denote this minimum value, the condition of stability is that

$$H < H_0.$$

25. The third method is the energy method, and the condition of stability is that the potential energy in the configuration of equilibrium should be a minimum. If G be the resultant flexural couple, A and C the flexural and torsional rigidities,* the value of the potential energy per unit of length is

$$W = \tfrac{1}{2}(G^2/A + H^2/C).$$

* The cross-section of the wire is supposed to be circular. There does not appear to be much advantage in taking into account any deviation from circularity in wires ordinarily met with. When the wire is a flat one, resembling a clock-spring, the theory of thin plates is more applicable.

Let $W+W'$, $G+G'$, $H+H'$ be the values of the potential energy and the flexural and torsional couples in any slightly displaced configuration; then

$$W' = \tfrac{1}{2}(G'^2/A + H'^2/C + 2GG'/A + 2HH'/C).$$

Now G' and H' are proportional to the small changes of curvature and twist which occur in passing from the equilibrium to the disturbed configuration, and these quantities can be found in terms of the displacements by means of the formulæ given by Mr. Love on p. 168 of his book; but in order to apply the energy method it is essential that we should know the correct expression for the potential energy to the *second* order of small quantities, whereas Mr. Love's formulæ only give the correct values of the product terms in W' to the first order. This method would therefore require us to calculate the changes of curvature and twist to the second order of small quantities; and this has not yet been done.

Stability of a Straight Wire subjected to Thrust.

26. The stability of a straight wire which is subjected to thrust is discussed in (amongst other places) Mr. Love's Treatise on Elasticity; but as his investigation does not bring out at all clearly the precise nature of the terminal conditions, I shall consider the subject afresh.

As there is no torsional couple, it will be sufficient to treat the problem as one of two dimensions. From the argument in §25, it follows that when the pressure or thrust at the extremities of the wire is sufficiently great, the wire will begin to bend and to assume the form of a curve of the elastica family, and we have to find the value of the thrust which is just sufficient to produce this state of things.

The equations of equilibrium of the deformed wire are

$$\frac{dT}{ds} - N\varpi = 0, \tag{1}$$

$$\frac{dN}{ds} + T\varpi = 0, \tag{2}$$

$$A\frac{d\varpi}{ds} + N = 0, \tag{3}$$

since $G = A\varpi$. From (1) and (3) we get

$$T = -P - \tfrac{1}{2}A\varpi^2,$$

as before; and since the curvature is a very small quantity, we may in (2) put $T = -P$, where P is the thrust applied to the ends of the wire. Whence by (2) and (3),

$$A \frac{d\varpi^2}{ds^2} + P\varpi = 0,$$

the integral of which is

$$\varpi = C \cos \mu s + D \sin \mu s, \quad \mu^2 = P/A. \tag{4}$$

27. Case I. Let the lower end A of the wire be firmly clamped, whilst the upper end B is pressed vertically downwards by a force P, but is otherwise free; also let l be the length of the wire, and let the arc s be measured from A.

At the end B, G and therefore ϖ are zero; whence

$$C \cos \mu l + D \sin \mu l = 0. \tag{5}$$

Also by considering the equilibrium of the whole wire, it follows that $N = 0$ at A, whence by (3) $d\varpi/ds = 0$ when $s = 0$, accordingly $D = 0$. This requires that $\cos \mu l = 0$, whence $\mu l = (2n + 1) \frac{1}{2} \pi$, or

$$P = \tfrac{1}{4} \pi^2 (2n + 1)^2 A/l^2. \tag{6}$$

The least value of the right-hand side of (6), which occurs when $n = 0$, gives the thrust P which must be applied to the upper end of the wire to produce an infinitesimally small deflection; if, therefore, the thrust is less than this quantity, no deflection will take place and the wire will remain straight. Whence the condition of stability is that

$$P < \tfrac{1}{4} \pi^2 A/l^2. \tag{7}$$

28. Case II. Let the wire be pressed between two parallel planes which are perpendicular to its undisplaced position. If the planes were perfectly hard, smooth and rigid (a condition which can only be approximately realized in nature), the ends of the wire would tend to slip on the slightest pressure being applied; we shall therefore suppose that the ends are in contact with mechanical appliances which will prevent any such slipping taking place, but are otherwise free.

Under these circumstances the terminal conditions are $\varpi = 0$ when $s = 0$ and $s = l$. Whence by (4)

$$C = 0, \quad \sin \mu l = 0, \quad \mu l = \pi,$$

and the condition of stability is that

$$P < \pi^2 A/l^2. \tag{8}$$

29. Case III. Let both ends of the wire be clamped. The terminal conditions require that the values of N and ϖ at the two ends should be equal to one another. Consequently

$$D(1 - \cos \mu l) = -\, C \sin \mu l,$$
$$C(1 - \cos \mu l) = \quad D \sin \mu l.$$

Eliminating C and D we obtain

$$\sin^2 \tfrac{1}{2} \mu l = 0, \quad \tfrac{1}{2} l = \pi,$$

and the condition of stability is

$$P < 4\pi^2 A/l^2. \tag{9}$$

The value of ϖ may now be written

$$\varpi = C \cos 2\pi s/l,$$

which shows that there are two points of inflexion, which occur when $s = \frac{1}{4} l$ and $s = \frac{3}{4} l$.

The first case corresponds to a column or pillar whose lower end is cemented into a bed of concrete, whilst the upper end supports a building which simply rests upon but is not fastened to the pillar; and we see that in this case the weight required to cause the pillar to collapse is less than in the other two cases. The second case corresponds to a pillar or rod both of whose ends rest on bearings to which they are not cemented. The third case corresponds to a pillar whose ends are respectively cemented to the foundations and the building supported. In the third case, the force required to cause the pillar to collapse is four times greater than in the second and sixteen times greater than in the first case.

Stability of a Straight Wire under Thrust and Twist.

30. We shall now suppose that a torsional couple is applied to the ends of the wire as well as a thrust, and shall investigate the conditions of stability.

In Case I, ϖ vanishes at one end because there is no flexural couple there; in Case II it vanishes at both ends; whilst in Case III the tangents at both extremities are parallel, and consequently there are at least two points of inflexion. Now from (16) of §17, and (23) of §21, it follows that ϖ can never

vanish unless the constant β which occurs in these equations is zero; accordingly the constant Q which appears in (12) of §16 must be zero.

The curvature is given by (10.A) of §15, in which Q must be put equal to zero and P to $-P$; also the term $\frac{1}{2}A^2\varpi^3$ being of the third order must be neglected and the equation becomes

$$\frac{d^2\varpi}{ds^2} + \left(\frac{H^2}{4A^2} + \frac{P}{A}\right)\varpi = 0, \tag{1}$$

where P now denotes the thrust; also by (8) and (10) of §15,

$$N_1 = -A\frac{d\varpi}{ds}, \tag{2}$$

$$N_2 = \tfrac{1}{2}H\varpi. \tag{3}$$

We have therefore

$$\varpi = C\cos\mu s + D\sin\mu s, \tag{4}$$

where

$$\mu^2 = \frac{H^2}{4A^2} + \frac{P}{A}. \tag{5}$$

Case I. At the end B, $\varpi = 0$, whence

$$C\cos\mu l + D\sin\mu l = 0, \tag{6}$$

whilst at the end A the shearing stress N_1, which is along the principal normal, must also vanish, which gives $D = 0$. Consequently the condition of stability is

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{4l^2}. \tag{7}$$

Case II. Here $\varpi = 0$ when $s = 0$ and $s = l$, when the condition will be found to be

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{l^2}. \tag{8}$$

Case III. In this case symmetry requires that the values of ϖ and N_1 should be equal when $s = 0$ and $s = l$. This leads to the condition

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{4\pi^2}{l^2}. \tag{9}$$

All these results agree with our former ones, as can be seen by putting $H = 0$.

Equilibrium and Stability of a Naturally Straight Wire deformed into a Helix.

31. It has been well known for many years that a helix is a possible figure of equilibrium for a naturally straight wire which is twisted as well as bent.

This result may easily be deduced by means of the general equations of equilibrium (1) of §2. In the helix

$$\frac{1}{\rho}=\frac{\cos^2\alpha}{a},\quad \frac{1}{\sigma}=\frac{\sin\alpha\cos\alpha}{a}, \tag{1}$$

where α is the pitch and a is the radius of the cylinder on which the helix is traced; also in the helix the principal normal is the normal to the cylinder.

Since the natural form of the wire is straight,

$$H=\text{const.}\quad G_1=0,\quad G_2=A/\rho; \tag{2}$$

also none of the quantities can be functions of s, whence it follows from the general equations that

$$N_1=0,\quad T=\frac{H}{\sigma}-\frac{A}{\sigma^2},\quad N_2=\frac{H}{\rho}-\frac{A}{\rho\sigma}. \tag{3}$$

These equations combined with (1) give

$$\left.\begin{aligned} T&=\frac{H\sin\alpha\cos\alpha}{a}-\frac{A\sin^2\alpha\cos^2\alpha}{a^2},\\ N_2&=\frac{H\cos^2\alpha}{a}-\frac{A\sin\alpha\cos^3\alpha}{a^2},\\ G_2&=\frac{A\cos^2\alpha}{a},\end{aligned}\right\} \tag{4}$$

whence

$$\left.\begin{aligned} &T\cos\alpha-N_2\sin\alpha=0,\\ &T\sin\alpha+N_2\cos\alpha=\frac{H}{a}\cos\alpha-\frac{A\sin\alpha\cos^2\alpha}{a^2}.\end{aligned}\right\} \tag{5}$$

Equations (5) show that the resultant force F which must be applied to the ends of the wire must be parallel to the axis of the cylinder on which the helix is traced, and that its magnitude is

$$F=\frac{H}{a}\cos\alpha-\frac{A}{a^2}\sin\alpha\cos^2\alpha. \tag{6}$$

The resultant couple $\mathfrak{G}$ is

$$\mathfrak{G}^2=H^2+\frac{A^2}{a^2}\cos^4\alpha. \tag{7}$$

The resultant force and couple are therefore to a certain extent arbitrary, since both contain the torsional couple H, the only limitation on whose value is

that it must not be large enough to break the wire or to produce a permanent set. We have therefore two special cases to consider.

32. Case I. Let $H = 0$; then the terminal stresses consist of a pushing force or thrust P, whose value is

$$P = A/a^2 . \sin\alpha \cos^2\alpha ,$$

together with a flexural couple G_2, whose value is $A/a . \cos^2\alpha$. The pitch of the helix is $\sin^{-1}(Pa/G_2)$, from which we see that in order that equilibrium may be possible Pa must not be greater than G_2.

33. Case II. Let $F = 0$, then

$$H = \frac{A}{a}\sin\alpha\cos\alpha = \frac{A}{\sigma}, \tag{8}$$

whilst $\mathfrak{G} = A/a . \cos\alpha$. The torsional couple is therefore proportional to the tortuosity; also since

$$H\cos\alpha - G_2\sin\alpha = 0,$$
$$H\sin\alpha + G_2\cos\alpha = A/a . \cos\alpha,$$

it follows that the terminal stress consists of a couple whose axis is parallel to the axis of the cylinder on which the helix is traced, and whose magnitude is $A/a . \cos\alpha$.

34. We shall now suppose that the wire is bent and twisted into a helix and the ends firmly clamped; and we shall investigate the condition that the helical form may be stable.

Let the wire be slightly displaced from its equilibrium configuration, and let $\rho^{-1} + p$ be its curvature. Substituting this value of ϖ in (10.A) of §15, we obtain

$$\frac{d^2p}{ds^2} + \left(\frac{H^2}{4A^2} - \frac{P}{A} + \frac{3Q^2\rho^4}{A^2} + \frac{3}{2\rho^2}\right)p = 0, \tag{9}$$

which determines the small change of curvature.

By (3), (6) and (9) of the present article, we obtain

$$P = \frac{H}{\sigma} - \frac{A}{\sigma^2} + \frac{A}{2\rho^2},$$

$$Q = \left(\frac{A}{\sigma} - \tfrac{1}{2}H\right)\frac{1}{\rho^2},$$

consequently (9) becomes

$$\frac{d^2p}{ds^2}+\left\{\left(H-\frac{2A}{\sigma}\right)^2+\frac{A^2}{\rho^2}\right\}\frac{p}{A^2}=0, \tag{10}$$

the solution of which is

$$p=C\cos\mu s+D\sin\mu s, \tag{11}$$

where

$$A^2\mu^2=\left(H-\frac{2A}{\sigma}\right)^2+\frac{A^2}{\rho^2}. \tag{12}$$

Let R, Θ, Z be the stresses along and perpendicular to the radius and parallel to the axis of the cylinder upon which the helix is traced; then

$$\left.\begin{aligned} N_1 &= -R,\\ T\cos\alpha-N_2\sin\alpha &= \Theta,\\ T\sin\alpha+N_2\cos\alpha &= Z.\end{aligned}\right\} \tag{13}$$

Since these equations are true in the case of the helical and the disturbed configuration, they will also be true when the variations of the stresses are substituted for their original values, in which case we have from (6), (8) and (10) of §15,

$$\left.\begin{aligned} T &= -\frac{Ap}{\rho},\\ N_1 &= -A\frac{dp}{ds},\\ N_2 &= (\tfrac{1}{2}H+Q\rho^2)p.\end{aligned}\right\} \tag{14}$$

Let us now suppose that the two ends of the wire lie on the same generator of the cylinder, so that the wire forms an even number of complete convolutions. From (13) and (14) the terminal conditions give

$$\left(\frac{dp}{ds}\right)_0=\left(\frac{dp}{ds}\right)_l,$$

$$p_0=p_l,$$

which by (11) become

$$-D(1-\cos\mu l)=C\sin\mu l,$$

$$C(1-\cos\mu l)=D\sin\mu l,$$

whence, eliminating C and D, we get

$$\sin^2\tfrac{1}{2}\mu l=0,$$

or

$$\mu l=2\pi.$$

Using the value of μ given by (12), we get

$$\left(H-\frac{2A}{\sigma}\right)^2+\frac{A^2}{\rho^2}=\frac{4\pi^2A^2}{l^2}. \tag{15}$$

Now (6) may be written

$$F\sin\alpha=\frac{H}{\sigma}-\frac{A}{\sigma^2}, \tag{16}$$

accordingly by (15) the condition of stability becomes

$$\frac{\mathfrak{G}^2}{A^2}-\frac{4F\sin\alpha}{a}<\frac{4\pi^2}{l^2}. \tag{17}$$

Since the ends of the wire are supposed to lie on the same generator of the cylinder, there must be m convolutions; whence the pitch of the helix is determined by the equation

$$2m\pi a\sec\alpha=l, \tag{18}$$

where m is an integer.

From (17) and (1) the condition may be written

$$\left(H-\frac{2A}{\sigma}\right)^2<\frac{4A^2\pi^2}{l^2}(1-m^2\cos^2\alpha),$$

which is impossible unless $\sec\alpha>m$.

When there is only one convolution $m=1$ and the condition becomes

$$H<\frac{6\pi A}{l}\sin\alpha, \qquad \alpha>0.$$

35. We shall now consider the two special cases.

In Case I the helix is held in equilibrium by a flexural couple and a thrust, and the condition (17) becomes

$$\cos^2\alpha\,(1+3\sin^2\alpha)<4\pi^2a^2/l^2<\cos^2\alpha/m^2,$$

or

$$1+3\sin^2\alpha<m^{-2},$$

which is impossible, and therefore the equilibrium is unstable.

In Case II the wire is held in equilibrium by a couple whose value is $A/a.\cos\alpha$, and the condition becomes $m<1$, which is impossible.

It therefore follows that in the two special cases the wire is unstable when it makes one complete convolution. When the wire does not make a complete convolution, the terminal conditions, and consequently the conditions of stability, will be represented by a different set of equations; but the investigation of the various cases which arise may be left to the reader.

Equilibrium and Stability of Circular Wire.

36. We shall now consider a problem concerning the stability of a naturally curved wire which covers a good many special cases.

A wire whose natural form is a tortuous curve is first unbent; secondly, the wire is twisted, and thirdly, the ends are joined together; it is required to find the condition that the circle is a possible figure of equilibrium, and that the circular form may be stable.

When the circle is a figure of equilibrium none of the quantities can be functions of s; we therefore obtain from (1) of §2,

$$\left.\begin{array}{lll} T=0 \quad , & N_1=0 \quad , & G_1=0 \quad , \\ H=\text{const.} & G_2=\text{const.} & N_2=H/a. \end{array}\right\} \tag{1}$$

The constancy of G_2 and H requires that the changes of curvature and twist which occur in passing from the natural to the circular form shall be constant quantities. These conditions will be satisfied if the natural form of the wire is a helix, which includes as a particular case a circular coil of *fine* wire, the radius of whose cross-section is small in comparison with the mean radius of the coil.

37. To investigate the stability we shall employ the second method. In the circular and the sinuous forms respectively let $H=C\tau$, and $H+H'=C\tau'$ be the torsional couples; then $H'=C(\tau'-\tau)$ where $\tau'-\tau$ is the small change of twist which occurs in passing from the circular to the sinuous form. This is a small quantity which can be expressed in terms of the four displacements u, v, w and β which the wire experiences in passing from one form to the other, whence by the third of equations (38) on p. 168 of Mr. Love's book,

$$H'=\frac{C}{a}\left(\frac{d\beta}{d\phi}+\frac{1}{a}\,\frac{dv}{d\phi}\right). \tag{2}$$

Let G_1', G_2+G_2' be the flexural couples when the wire is sinuous; then from the same equations it follows that

$$\left.\begin{array}{l} G_1'=\dfrac{A}{a}\left(\beta-\dfrac{1}{a}\,\dfrac{d^2v}{d\phi^2}\right), \\ G_2'=\dfrac{A}{a^2}\left(\dfrac{d^2u}{d\phi^2}+u\right) \quad , \end{array}\right\} \tag{3}$$

since the wire is supposed to be inextensible.

The radius of torsion of the sinuous curve is given by equation (23) on p. 163 of Love's book, and in the present case is

$$\frac{1}{\sigma'} = \frac{1}{a^2}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right). \tag{4}$$

The equations of equilibrium when the wire is slightly sinuous are now

$$\left.\begin{aligned} \frac{dT}{d\phi} - N_1 &= 0\,, \\ \frac{dN_1}{d\phi} - \frac{H}{a^2}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right) + T &= 0\,, \\ \frac{dN_2'}{d\phi} \qquad &= 0\,, \\ \frac{dH'}{d\phi} - G_1' &= 0\,, \\ \frac{dG_1'}{d\phi} - \frac{G_2}{a}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right) + \frac{H}{a}\left(\frac{d^2u}{d\phi^2} + u\right) + H' - N_2'a &= 0\,, \\ \frac{1}{a}\,\frac{dG_2'}{d\phi} + N_1 &= 0\,. \end{aligned}\right\} \tag{5}$$

Let $D = d/d\phi$; then by (2) and (3) the fourth of (5) becomes

$$a\,(CD^2 - A)\,\beta + (A + C)\,D^2 v = 0. \tag{6}$$

Differentiating the fifth and taking into account the fourth of (16) and also (2) and (3) we get

$$(A + C)\,D^2\beta + a^{-1}\{C - AD^2 - aG_2\,(D^2 + 1)\}\,D^2 v + H\,(D^2 + 1)\,Du = 0. \tag{7}$$

Eliminating T and N_1 between the first, second and sixth, we get

$$Ha\,(D^2 + 1)\,D^2 v + A\,(D^2 + 1)^2\,Du = 0. \tag{8}$$

Since the circle is *complete,* all the quantities must be functions of $\varepsilon^{\iota s\phi}$, where s is any integer greater than unity, since $s = 0$ and $s = 1$ correspond to rigid body displacements which can produce no alteration in the state of strain. Our equations accordingly become

$$\left.\begin{aligned} &a\,(Cs^2 + A)\,\beta + (A + C)\,s^2 v = 0\,, \\ &(A + C)\,s\beta + a^{-1}\{C + As^2 + aG_2\,(s^2 - 1)\}sv + \iota H\,(s^2 - 1)\,u = 0\,, \\ &Hasv + \iota A\,(s^2 - 1)\,u = 0\,. \end{aligned}\right\} \tag{9}$$

Eliminating u, v and β we get

$$H^2a^2 = \left\{ G_2 a + \frac{AC(s^2-1)}{A+Cs^2} \right\} A(s^2-1). \tag{10}$$

If, therefore,

$$H^2a^2 < \left\{ G_2 a + \frac{AC(s^2-1)}{A+Cs^2} \right\} A(s^2-1), \tag{11}$$

it will be impossible to satisfy the conditions of equilibrium, consequently (11) is the condition of stability.

If the curvature is increased by deformation, G_2 is a positive quantity, and the least value of the right-hand side of (11) occurs when $s=2$; under these circumstances the condition becomes

$$H^2a^2 < 3A \left\{ G_2 a + \frac{3AC}{A+4C} \right\}. \tag{12}$$

But if, on the other hand, the curvature is diminished by deformation, G_2 will be negative, and the inequality (12) involves the subsidiary condition that

$$G_2 a + \frac{3AC}{A+4C} \tag{13}$$

should be positive.

38. Before discussing the general condition, it will be desirable to consider the subsidiary condition (13). Let ρ be the radius of curvature of the undeformed wire, and let $\rho < a$; then

$$-G_2 a = A\left(\frac{a}{\rho} - 1\right),$$

and the condition becomes

$$\frac{a}{\rho} - 1 < \frac{3C}{A+4C}.$$

For metal wires $q/n = \frac{5}{2}$ about, so that $A/C = \frac{5}{4}$, and the last equation becomes

$$a < \tfrac{11}{7}\rho. \tag{14}$$

It therefore follows that the circular form will be unstable if its radius is greater than $\frac{11}{7}$ths of the radius of principal curvature of the undeformed wire.

When the natural form of the wire consists of a circular coil which is unrolled and the ends joined together, the preceding result shows that the circular form will be unstable when the length of the coil is greater than about one

and a half complete convolutions. We shall presently consider what will happen when this length is exceeded.

39. When the natural form of the wire consists of a *complete* circle of radius a, which is cut and then twisted and its ends joined together, $G_2 = 0$ since the twist produces no change of curvature. Under these circumstances the condition (12) becomes

$$Ha < 3A\sqrt{\frac{C}{A+4C}}.$$

Assuming that $A/C = \frac{5}{4}$, this condition requires that the total twist should not be greater than $\pi \times 3.27$; that is, about six and a half right angles.

Let the natural form of the wire be a helix of pitch α, and let $2\pi l$ be the length of a complete convolution. Then $l\cos\alpha$ is the radius of the cylinder upon which the helix is traced, and $l\sec\alpha$ is its radius of curvature; whence the subsidiary condition (13) becomes

$$a\cos\alpha < \tfrac{11}{7}l;$$

that is to say, the projection of the length of the wire upon a circular section of the cylinder must not be greater than $\frac{11}{7}$ths of a complete convolution.

When the above condition is satisfied, the circular form will be stable when a torsional couple is applied to the ends of the wire before they are soldered together, provided the twist does not exceed a certain magnitude which is determined by (12).

40. We shall now consider the period equation when a complete circular wire is performing small oscillations about its configuration of stable equilibrium. The method employed is precisely similar to the investigation given on p. 121 of my paper on wires,* and equation (50) on p. 122 is a particular case of the more general result which we shall proceed to consider. I find that the periods are given by the following cubic equation:

$$\begin{aligned}
2\pi^2c^4a^6\Big[&a^2(1+\tfrac{1}{4}\lambda^2s^2)h^2p^4 \\
&-\tfrac{1}{2}\{q+2ns^2+\tfrac{1}{4}\lambda^2s^2(q+2qs^2+4n+2ns^2)+\tfrac{1}{4}\lambda^2s^2(s^2-1)qR\}hp^2 \\
&+\frac{\lambda^2qs^2(s^2-1)}{4a^2}\{n(s^2-1)+\tfrac{1}{2}(q+2ns^2)R\}\Big] \\
&\qquad\times\Big[\{s^2+1+\tfrac{1}{4}\lambda^2(s^2-1)^2\}hp^2-\frac{\lambda^2qs^2(s^2-1)^2}{4a^2}\Big] \\
&+H^2s^4(s^2-1)^2(q+2ns^2-2a^2hp^2)=0. \qquad (15)
\end{aligned}$$

* Proc. Lond. Math. Soc., Vol. XXIII.

In this equation c is the radius of the cross-section $\lambda = c/a$, q and n are Young's modulus and the rigidity, h the density, p the period, and $R = 1 - a/\rho$, where ρ is the radius of curvature of the natural form.

Omitting superfluous positive factors, the term independent of p is

$$H^2a^2 - \tfrac{1}{8}\pi^2q^2c^8(s^2-1)\left\{\frac{n(s^2-1)}{q+2ns^2} + \tfrac{1}{2}R\right\},$$

and the condition that one of the roots of the cubic should be real and positive is that this quantity should be negative. This condition is easily seen to be equivalent to (11).

To investigate the conditions that the remaining roots should be real and positive would be a somewhat troublesome operation; but there can be little doubt that the conditions of stability already given are sufficient to insure that this should be the case.

41. We shall now consider the case in which there is no twist. Under these circumstances the cubic splits up into two factors, the second of which gives the periods of the vibrations of a Hoppe's ring, whilst the first factor leads to an equation equivalent to (50) of my former paper, with which it becomes identical when $R = 0$, as was the case in the problem there considered. We therefore see that the vibrations consist of two distinct types, viz. flexural vibrations in the plane of the ring, and vibrations which involve torsion and flexion perpendicular to this plane. The periods of the purely flexural vibrations are always real, and consequently the ring is stable for displacements in its own plane; but if R is a negative quantity whose numerical value is greater than the least value of $2n(s^2-1)/(q+2ns^2)$, the absolute term of the first factor will be negative and the motion will be unstable. This leads to the subsidiary condition (13).

From these results we see that when the circular form becomes unstable the ring will not collapse like a boiler flue, but will assume the form of a bent and twisted tortuous curve. They also to a certain extent show what the form of this curve will be. Assuming that $A/C = \frac{5}{4}$, it follows that the value of

$$\frac{C(s^2-1)}{A+Cs^2}$$

when $s = 2$ is $\frac{4}{7}$, and its value when $s = \infty$ is unity. If therefore G_2a/A has a negative value whose numerical value lies between $\frac{4}{7}$ and 1, a sinuous figure differing slightly from a circle will be possible; but if this numerical value

is considerably greater than unity, a sinuous form will be impossible and the unstable circle will make a sudden jump, and will assume the form of some entirely different curve or may even turn itself inside out.

42. A great many other special problems can be solved by the above methods; but when a wire whose natural form is a straight line is twisted and the ends soldered together, the condition of stability cannot be obtained by means of the general formula (11). The problem is one of those special cases which so frequently occur in mathematics in which a formula of apparent generality fails to give a correct result in some particular instance and a procedure of a special kind has to be resorted to. When the wire is naturally straight H' is constant and G_1' is zero, so that the fourth of (5) disappears as well as the term $dG_1'/d\phi$ in the fifth equation. Under these circumstances the first of (9) disappears whilst the second becomes, since $G_2 a = A$,

$$Hau - \iota Asv = 0.$$

The third of (9) remains unaltered, so that we get

$$H^2a^2 = A^2(s^2 - 1),$$

which gives

$$\tau = \frac{q\sqrt{3}}{2na}$$

as the condition of stability.

This result appears to have been first given by Mr. Michell,* who obtained it by supposing the wire to perform small oscillations. Assuming that $q/n = \frac{5}{2}$, he found that the total twist must be less than $2\pi \times 2.16$, that is, less than eight and a half right angles.

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* Mess. Math., Vol. XIX, p. 184.